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Gravitational energy of core evolution: implications for thermal history and geodynamo power

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Abstract

Using density–pressure relationships for mantle silicate and core alloy closely matching PREM we have constructed six models of the Earth in different evolutionary states. Gravitational energies and elastic strain energies are calculated for models with homogeneous composition, separated mantle and liquid core, separated inner and outer cores with the inner core either liquid or solid and models with increased densities, representing cooling of either the mantle or core. In this way we have isolated the gravitational energy released by each of several evolutionary processes and subtracted the consequent increase in strain energy to obtain the net energy released as heat or geodynamo power. Radiogenic heat ($\sim 7.8 \times 10^{30}$ J) is found to contribute only about 25% of the total heat budget, the balance originating as residual gravitational energy from the original accretion and from core separation (14×10^{30} J). The total energy of compositional convection, driven by inner core formation, is 3.68×10^{28} J and this is the most important (or even the only) energy source for the dynamo for the most recent 2 billion years. It appears unlikely that the inner core existed much before that time. The total net (gravitational minus strain) energy released in the core by the process of inner core formation, 11.92×10^{28} J, is not much less than the thermal energy released in this process, 15.1×10^{28} J. In the mantle the net (gravitational minus strain) energy released by thermal contraction is about 20% of the heat release. All of the numerical results are presented in a manner that allows simple rescaling to any revised density estimates.

Keywords: Inner core formation; Mantle; Density–pressure relationship

1. Introduction

The pressure at the centre of the Earth is about 364 GPa. For a sphere of uniform density with the same mass, M, and radius, R, the central pressure would be

$$P_0 = \frac{3}{8\pi}\frac{GM^2}{R^4} = 174 \text{ GPa} \tag{1}$$

where $G = 6.6726 \times 10^{-11}$ m^3 kg^{-1} s^{-2} is the gravitational constant. Eq. (1) makes no allowance for the increase in density with depth by self-com-

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pression. As reported here, for a homogenised Earth in which self-compression is allowed for, the central pressure would be only about 215 GPa. Thus most of the increase in central pressure, relative to a uniform sphere, is caused by the separation of the dense core. An intuitive conclusion is that the increase in compression and consequent contraction releases gravitational energy additional to that calculated simply by redistributing material without allowing for the increase in compression.

Core formation energy may be regarded as a late component of the gravitational energy of accretion of the Earth. Its importance to early history and thermal evolution is considered in Section 4. However, core evolution is a continuing process and, as Loper (1978a) recognised, the effect of increasing compaction is relevant to the energy source for the dynamo as well as to the global energy balance. Braginsky (1963) first pointed out that growth of the inner core by progressive freezing of outer core material releases at the inner core boundary light solutes that are rejected by the solid and provide a source of buoyancy for stirring the outer core. Following studies by Gubbins (1977) and Loper (1978a,b), this became recognised as the dominant (or sole) source of power for the geomagnetic dynamo. However, doubt has been expressed (e.g., by Buffett et al., 1996) that it is adequate without some admixture of thermally driven convection and the problem needs re-examination.

Several authors have addressed this issue. Häge and Müller (1979) emphasised the importance of gravitational energy to inner core freezing, but did not consider compositional separation. Gubbins et al. (1979) and Buffett et al. (1996) approached the problem in a general way, but it has not been clear how different the effects of mantle compression and strain energy generation are for the processes of compositional separation, freezing and general cooling of the mantle as well as the core. We have aimed to isolate these effects clearly in a series of numerical models.

An effect which we have chosen not to emphasise is adiabatic heating, which we regard as a component of elastic strain energy, arising from the difference between adiabatic and isothermal incompressibilities. It cannot cause convection because an adiabatic medium that is compressed remains adiabatic: no convective energy is generated, but only a slight increase in conducted heat. We consider it better not to regard adiabatic heating as an energy source, but, for completeness of the record, adiabatic heating is added as a final column of Table 5.

The calculation of Loper (1978a) assumed a much larger density contrast between the inner and outer cores (2630 kg m^{-3}) than is now recognised (e.g., 597.26 kg m^{-3} in PREM—Dziewonski and Anderson, 1981) and so led to an overestimate of the available dynamo energy. If this were the only problem, then Loper's calculation could simply be rescaled, but there are three further problems that can be addressed only by a new calculation.

(1) The total gravitational energy release is not a satisfactory measure of the energy available to the dynamo. Some of the energy derived from inner core formation is released in the mantle. Conversely, cooling and shrinking of the mantle compresses the core, releasing gravitational energy there also. Energies in the core and mantle must be estimated separately, taking account of the effects of each on the other.

(2) The effects of compositional separation and freezing of the inner core are quite different. Freezing causes a much greater global contraction. From our results (summarised in Tables 2 and 5), a solidification density increment of 140 kg m^{-3} causes a 0.319 km decrease in the Earth's radius. By contrast, mass redistribution of compositional separation resulting in a 457 kg m^{-3} inner core–outer core density contrast causes only a 0.028 km contraction. Normalised to the same density contrast, freezing releases 3.2 times as much gravitational energy, but it can contribute to the dynamo only with the modest thermodynamic efficiency of thermal energy. We have greater interest in the thermodynamically very efficient compositional convection, but its gravitational effect is weaker.

(3) Part of the gravitational energy release appears as elastic strain energy, which is stored irretrievably in the Earth and must be subtracted from the energy release in calculating contributions to either dynamo power or heating (apart from adiabatic compression).

We have examined these problems by considering six models of a progressively evolving Earth:

Model A—a compositionally uniform Earth, with core, mantle and crustal material homogenised, subject to self-compression and including phase changes in the silicate.

Model B—a compositionally uniform liquid core surrounded by a solid silicate mantle.

Model C—a separate, denser inner core, but still liquid.

Model D—a solidified inner core. Comparison of models B, C and D allows the effects of compositional separation and freezing to be calculated separately, even though they are not physically separable processes.

Model E—model D but with an overlying mantle 1.01 times as dense. This is a crude simulation of cooling of the mantle by about 500 K.

Model F—model D but with core density multiplied by 1.01 throughout, approximating the effect of core cooling by 660 K. This is about ten times the cooling believed to accompany inner core formation.

None of these models has a crust, but the silicate density is adjusted downwards, relative to PREM, to simulate a homogenised crust–mantle mix.

2. Equations of state and a pressure–density table

The earth model PREM (Dziewonski and Anderson, 1981) was used as the starting model for the density–pressure relationships of mantle silicate and core alloy. The homogenised model requires extrapolation of mantle densities into the core pressure range and alloy densities to zero pressure and for this purpose we need equations of state. These two extrapolations present different problems.

In the case of mantle silicate, conventional finite strain theory (e.g., Birch, 1952) is unsatisfactory because the extrapolation extends well beyond the range of observations and the Birch theory cannot be adjusted to match the high pressure asymptote of $K' = \mathrm{d}K/\mathrm{d}P$ for the lower mantle, where K is incompressibility and P is pressure. This problem was addressed by Stacey (1998) who argued that the obvious solution is to write an equation of state in terms of the dimensionless quantities K' and P/K. Two equations of this form were found to give excellent fits of the PREM lower mantle tabulations of K, density, ρ, and P when constrained to the value of $K'_\infty = (\mathrm{d}K/\mathrm{d}P)_{P\to\infty}$ obtained by fitting the lower mantle data to the μ–K–P equation

$$\frac{\mu}{K} = \left(\frac{\mu}{K}\right)_0 \left(1 - K'_\infty \frac{P}{K}\right) \tag{2}$$

where μ is rigidity. We refer to the first of these equations-of-state as the quadratic K'–P/K equation

$$K' = K'_0 + K'_0 K''_0 P/K + \frac{1}{2} K_0^2 K''_0 (P/K)^2 \tag{3}$$

where

$$\frac{1}{2} K_0^2 K'''_0 = K_0'^3 - K'_0 K_\infty'^2 - K_0 K''_0 K'_\infty \tag{4}$$

and the second as the exponential K'–P/K equation

$$K' = K'_0 (K'_\infty / K'_0)^{K'^{P/K}_\infty} \tag{5}$$

The zero pressure limit of K', $K'_0 = 4.21$ was chosen to match an appropriate mix of laboratory values for silicate perovskite and magnesiowustite.

We have chosen to use Eq. (5) for the lower mantle extrapolation. Fitted to PREM it gives a zero pressure density $\rho_0 = 3972.71$ kg m^{-3} (Eq. (3) gives $\rho_0 = 3971.74$ kg m^{-3}). The models considered in this paper assume that there is no crust and all mantle densities are adjusted downwards by the factor 0.99733 to allow for mixing with the intrinsically lighter crust. This factor corresponds to ideal mixing of 4.00×10^{24} kg of mantle with 3.125×10^{22} kg of (crust + ocean) with average density 0.7433 times the mantle density. Thus the zero pressure density for the lower mantle material used to calculate the silicate densities at pressures above 23.8 GPa by Eq. (5) is $\rho_0 = 3962.10$ kg m^{-3}. The corresponding zero pressure incompressibility is $K_0 = 203.60$ GPa. The formal uncertainty of the lower mantle extrapolation is $\Delta\rho_0 = \pm 5$ kg m^{-3}, but digits well beyond the level of absolute significance are retained in the calculations to ensure that no differences between our models arise from rounding errors. The silicate and composite density tabulations stop at 240 GPa, which is beyond the pressure to which they are compressed in any of the models.

Of course variations in temperature from the present Earth are implicitly disallowed in the models A to D. Loss of heat by the real, evolving Earth causes gravitational energy release by thermal contraction in addition to the energies calculated for models A to

D. Models E and F are included to see how significant this is.

At pressures below 23.8 GPa, that is for the upper mantle and transition zone, we have followed the PREM density tabulation over each of the identified pressure ranges, but with the factor 0.99733 to allow for the admixture of crustal material. The uppermost layer (pressure below 7.1 GPa) incorporates the density inversion that extends to 220 km depth in PREM. The physical reality of this, or any other fine details of the shallow density structure, are not important to our discussion because we use the same basic density–pressure table for all models.

An equation-of-state for core material extending to zero pressure is more problematic. The laboratory value of K_0' for liquid iron (4.66, Anderson and Ahrens, 1994) cannot be integrated to K' for the outer core by any plausible equation without giving a substantial discrepancy with the increase in K over this pressure range. There must be structural changes in liquid iron, presumably corresponding to phase changes in the solid, but without more information it is not clear how the changes in elasticity should be modelled.

In tabulating alloy densities we have used Eq. (3) which appears more suitable than Eq. (5) because it requires an additional core datum to fit the constants and so is marginally less dependent on assumptions that are needed about K_0' and K_∞'. The value $K_\infty' = 2.8$ is adopted from a calculation by Stacey (1995), but is not critical because the required extrapolation is to lower pressures than in the PREM tabulation and not to higher pressures. More serious is the uncertainty in guestimates of K_0 and K_0'. There is a trade-off between these quantities and our choice was dictated by adjusting K_0' as necessary to obtain a realistic value of K_0 by integrating Eq. (3) to core pressures. We assumed $K_0' = 5.4$, $K_0 = 100$ GPa. This is equivalent to assuming that K_0' is very temperature-dependent and that the adiabatically extrapolated core temperature is about 2500 K. Such a strong temperature-dependence of K' is implausible, but the effect of this assumption on the core density extrapolation is similar to the assumption that over the pressure range 0 to 135 GPa there are distributed increments in K due to structural changes in liquid iron. With these assumptions, Eq. (3) requires $K_0 K_0'' = -10.455_{15}$ and $1/2 K_0^2 K_0'' = 8.89_{04}$. Then the outer core density tabulation of PREM extrapolates to $\rho_0 = 6370.75$ kg m^{-3}. While this appears to be in satisfactory agreement with other finite strain estimates of ρ_0 for the outer core (e.g., table 3 of Stacey, 1995), all of them have ignored the fundamental difficulty discussed here and all are therefore equally suspect.

It should be noted that only model A is affected by these major density extrapolations.

Eq. (3) was used to construct a table of 'raw' outer core densities over the whole pressure range in Table 1 and these were then adjusted in various ways to provide densities for the different models. The precision of these adjustments is crucial to validity of our calculations of the small energy differences between models B, C and D. For this purpose we need to estimate the density increment on solidification of core material with constant composition and then identify the difference between this increment and the density jump at the inner core boundary with the compositional difference between the inner and outer regions of the core.

To estimate the density decrement on melting we equated the differential Lindemann formula for the variation of melting point T_M with pressure

$$\frac{dT_M}{dP} = \frac{2\left(\gamma - \frac{1}{3}\right)T_M}{K_T} \tag{6}$$

where γ is the Grüneisen parameter and K_T is the isothermal modulus, to the Clausius–Clapeyron equation

$$\frac{dT_M}{dP} = \frac{\Delta V}{\Delta S} \tag{7}$$

ΔV and ΔS being the volume and entropy increments on melting, and appealed to the theory of entropy of melting of simple metals at high pressure, as outlined by Poirier (1991) (pp. 101–103). This gives two terms for the entropy of melting of n moles of a substance

$$\begin{aligned}\Delta S &= nR\ln 2 + \alpha K_T \Delta V \\ &= nR\ln 2 + \gamma\rho C_V \Delta V\end{aligned} \tag{8}$$

where R is the gas constant, α is volume expansion coefficient and C_V is specific heat (at constant vol-

Table 1
Pressure–density relationships for total silicate (homogenised mantle plus crust), core alloy and a composite with proportions as in the earth

Pressure (GPa)	Silicate density ($kg\ m^{-3}$)	Alloy density ($kg\ m^{-3}$)	Composite density ($kg\ m^{-3}$)
0	3373.66	6386.68	3983.89
7.1	3350.52	6785.13	4009.49
7.1	3426.60	6785.13	4082.75
13.3	3533.78	7070.54	4218.98
13.3	3713.83	7070.54	4390.60
23.8	3981.47	7476.10	4693.85
23.8	4361.37	7476.10	5043.62
30	4450.10	7684.46	5154.50
40	4584.10	7986.53	5319.96
50	4708.83	8256.84	5472.31
60	4825.92	8502.99	5614.19
70	4936.61	8730.01	5747.47
80	5041.80	8941.39	5873.50
90	5142.53	9139.68	5993.60
100	5238.44	9326.85	6107.71
110	5330.95	9504.39	6217.36
120	5420.12	9673.51	6322.76
130	5506.28	9835.20	6424.37
140	5589.71	9990.24	6522.54
150	5670.65	10139.33	6617.60
160	5749.29	10283.00	6709.80
170	5825.82	10421.77	6799.37
180	5900.39	10556.04	6886.53
190	5973.16	10686.19	6971.45
200	6044.22	10812.51	7052.44
210	6113.69	10935.30	7135.15
220	6181.68	11054.80	7214.21
230	6248.27	11171.24	7291.56
240	6313.55	11284.80	7367.30
250		11395.67	
260		11504.02	
270		11609.97	
280		11713.67	
290		11815.24	
300		11914.80	
310		12012.43	
320		12108.24	
330		12202.31	
340		12294.72	
350		12385.54	
360		12474.84	
370		12562.70	
380		12649.20	

ume). The first term in Eq. (8) is the configuration entropy of structural disorder caused by melting and the second term accounts for the entropy change of isothermal expansion. Equating Eqs. (6) and (7) and substituting for ΔS by Eq. (8) we have

$$\Delta V = (nR/K_T)\ln 2\left\{\left[2T_M\left(\gamma - \frac{1}{3}\right)\right]^{-1} - \alpha\right\}^{-1} \tag{9}$$

The first term in this equation is the important one. With numerical values appropriate to inner core boundary conditions (mean atomic weight 48.1—Stacey, 1992, $K_T = 1289$ GPa, $T_M = 5000$ K, $\gamma = 1.2$, $\alpha = 7 \times 10^{-6}\ K^{-1}$) and with density 12 763 kg m^{-3}, Eq. (9) gives

$$\Delta\rho\ (\text{freezing}) = 140\ \text{kg m}^{-3} \tag{10}$$

Combining this result with PREM densities at the inner core boundary we have

$$\left.\begin{aligned} \rho_1\ (\text{outer core}) &= 12\,166.34\ \text{kg m}^{-3} \\ \rho_2\ (\text{inner core, solid}) &= 12\,763.60\ \text{kg m}^{-3} \\ \rho_3\ (\text{inner core, melted}) &= 12\,623.60\ \text{kg m}^{-3} \end{aligned}\right\} \tag{11}$$

Assuming simple mixing, the corresponding density for a homogenised liquid core is

$$\bar{\rho} = \left[\frac{m_1/(m_1+m_2)}{\rho_1} + \frac{m_2/(m_1+m_2)}{\rho_3}\right]^{-1} = 12\,188.74677\ \text{kg m}^{-3} \tag{12}$$

where $m_2 = 9.8433 \times 10^{22}$ kg is the mass of the inner core and $(m_1 + m_2) = 1.939548 \times 10^{24}$ kg is the mass of the whole core obtained by integrating the PREM density parameters. The alloy densities listed in Table 1 are homogenised whole liquid core values obtained by dividing the 'raw' outer core density values given by Eq. (3) by the factor $\rho_1/\bar{\rho} = 0.998161684$. For model B, the alloy densities were used as tabulated. For models C, D and E, the outer core densities were multiplied by the factor $\rho_1/\bar{\rho}$. Inner core densities were multiplied by the factor $\rho_3/\bar{\rho} = 1.035676615$ for model C and by $\rho_2/\bar{\rho} = 1.04716262$ for models D and E. Model F multiplies both inner and outer core densities by 1.01, relative to model D, with no change in mantle density.

For model A the whole earth homogenised density tabulation was used. At each pressure this was obtained by averaging the listed alloy and silicate densities by weighting according to the core mass and total Earth mass (5.9737×10^{24} kg), in the manner of Eq. (12). It is only for model A that the major density extrapolations and the equations of state are required. The other models barely depart from the PREM pressure range for each material.

3. A comparison of six evolving earth models

Each model was calculated iteratively by starting with an assumed central pressure, P_0, and integrating outwards with mass accumulating as

$$M(r) = 4\pi\int_0^r \rho r^2 \mathrm{d}r \tag{13}$$

with gravity at each level given by

$$g = GM(r)/r^2 \tag{14}$$

and with pressure progressively decreasing as

$$P = P_0 - \int_0^r \rho g\,\mathrm{d}r \tag{15}$$

The integration stopped at $P = 0$ and the difference between the accumulated mass and the (target) total mass of the Earth (5.9737×10^{24} kg) was used to re-estimate the starting central pressure and the cycle was repeated until the residual mass discrepancy was less than 10^{-13} of the Earth's mass. Density was assumed to vary linearly with pressure over each of the pressure intervals in Table 1.

Density was adjusted discontinuously at each boundary. For model A the composite densities in Table 1 were used and the only boundaries are those due to silicate phase transitions at 23.8 GPa, 13.3 GPa and 7.1 GPa. Model B used the alloy density, as listed, until the core mass had accumulated and then the silicate densities. Models C, D, E and F accumulated an inner core of higher density with an outer core of lower density, as detailed in Section 2.

For each model the gravitational energy release was calculated by

$$E_G = G\int \frac{M(r)}{r}\mathrm{d}M = 4\pi\int g\rho r^3 \mathrm{d}r \tag{16}$$

This quantity was the principal target of the investigation. Part of it is consumed as elastic strain energy of self compression ($-\int P\mathrm{d}V$ for all mass elements)

$$E_E = \int M(r)P\frac{\mathrm{d}\rho}{\rho^2} = -\int \frac{M(r)gP}{\rho}\left(\frac{\mathrm{d}\rho}{\mathrm{d}P}\right)\mathrm{d}r \tag{17}$$

where $(-\mathrm{d}\rho/\mathrm{d}P)$ is given by $(\rho_2 - \rho_1)/(P_1 - P_2)$

Table 2
Numerical results for six models

Model	A	B	C	D	E	F
Feature	Homogeneous	Uniform core	Liquid inner core	Solid inner core	Cooled mantle	Cooled core
Radius (km)	6364.6253	6371.9449	6371.9166	6371.5978	6350.0492	6365.8770
Core radius (km)	–	3480.2297	3480.1576	3479.3457	3477.4487	3464.7414
Inner core radius (km)	–	–	1226.0571	1221.2748	1220.7620	1215.7971
Central pressure (GPa)	214.76152	359.42457	362.69161	363.96463	365.70619	369.15912
Inner core boundary pressure (GPa)	–	–	328.19051	328.91516	330.59869	333.47138
Core–mantle boundary pressure (GPa)	–	135.52332	135.52895	135.59230	136.91041	136.73782
I/MR^2	0.37272007	0.33123731	0.33117022	0.33111528	0.33163632	0.33046439
Gravitational energy (10^{30} J)	232.67485	248.67028	248.71178	248.75301	249.34348	249.28922
Elastic energy (10^{30} J)	13.66190	15.71694	15.72163	15.72678	15.80212	15.79985
Core gravitational energy (10^{30} J)	–	43.87086	43.91065	43.93266	43.95439	44.122862
Core elastic energy (10^{30} J)	–	8.30523	8.30970	8.31237	8.37340	8.34069
Depth to 23.8 GPa (km)	572.1198	667.6496	667.6428	667.5661	656.6123	666.1906

Table 3
Details of the homogeneous model of the Earth (Model A)

Radius (km)	Density (kg m^{-3})	Pressure (GPa)	Gravity (m s^{-2})
0	7172.7946	214.76152	0
500	7158.6060	212.96686	1.0012121
1000	7115.4401	207.61699	1.9952575
1500	7042.8527	198.81623	2.9745795
2000	6943.7026	186.73252	3.9325814
2500	6813.3148	171.59991	4.8615778
3000	6651.9016	153.72035	5.7536582
3500	6458.3247	133.45877	6.6007494
4000	6230.4625	111.24312	7.3941939
4500	5964.3557	87.56500	8.1244583
5000	5633.9919	82.98634	8.7803392
5500	5289.3561	38.15038	9.3481061
5792.5055	5043.6200	23.80000	9.6320658
5792.5055	4693.8500	23.80000	9.6320658
6000	4429.0405	14.63100	9.7434449
6030.9550	4390.6000	13.30000	9.7575543
6030.9550	4218.9800	13.30000	9.7575543
6183.7050	4082.7500	7.10000	9.7999584
6183.7050	4009.4900	7.10000	9.7999584
6364.6253	3983.8900	0	9.8399483

over each of the pressure ranges in Table 1. We also calculated the moment of inertia

$$I = \frac{2}{3}\int r^2 \mathrm{d}M = \frac{8}{3}\pi\int \rho r^4 \mathrm{d}r \qquad (18)$$

Results are detailed in Table 2. Table 3 gives the profile of the homogeneous model A which is the only model differing markedly from the present Earth.

Except for model A, the differences between models are quite small. Thus, in calculating gravitational energies we have sought small differences between large quantities and so have retained in the calculations digits that are not significant in absolute terms, but are needed in comparing models. The coupled integrals in Eqs. (13)–(18) were evaluated using The Mathworks' Simulink to call Matlab routine 'ODE45' (Shampine and Reichelt, 1997). This uses a pair of Runge–Kutta formulae of orders 4 and 5 (Dormand and Prince, 1980, 1986) to both solve the ODEs and control the relative precision, which we set to the minimum achievable, 2.8×10^{-14}, well below the numerical significance of any of the results.

4. The role of gravitational energy in early thermal history

The gravitational energy of accretion of the Earth dwarfs all other energy sources (Table 4). Although most of this energy must have been radiated away during the accretion process itself, without some of this original energy we could not explain the present high temperatures in the interior. The stored heat is more than twice the heat generation by uranium, thorium and potassium in the past 4.5×10^9 years and even the combination of radioactivity and core separation energy falls well short of the total heat loss plus stored heat.

The first three entries in Table 4 are taken from Table 2 by subtracting strain energy differences from gravitational energy differences. Radiogenic heat is calculated by integrating the decays of the four thermally important isotopes, ^{238}U, ^{235}U, ^{232}Th and ^{40}K, from $t = -\tau = -4.5 \times 10^9$ years to $t = 0$

$$Q = \dot{Q}_0 \sum_i \int_{-\tau}^{0} f_i \mathrm{e}^{-\lambda_i t} \mathrm{d}t = \dot{Q}_0 \sum_i \frac{f_i}{\lambda_i}(\mathrm{e}^{\lambda_i t} - 1) \qquad (19)$$

We assume that the present radiogenic heating rate is $\dot{Q}_0 = 28.9 \times 10^{12}$ W and that the fractions, f_i, of this total contributed by the four isotopes are those estimated by Stacey (1992) (p. 323). The stored heat assumes adiabatic temperature profiles, anchored to 1950 K at 670 km depth in the mantle and 5000 K at the inner core boundary. No reasonable adjustment of these numbers would significantly modify our conclusions unless there was intense early short-lived radioactivity, a point that we return to below.

Table 4
A comparison of energies in the earth. All values in units of 10^{30} J. The first three entries are net gravitational energy release with elastic strain energy subtracted

Accretion of a homogeneous mass	219.01
Core separation	13.94
Inner core formation	0.07289
Elastic strain energy	15.80
Radiogenic heat in 4.5×10^9 years	7.8
Residual (stored) heat	13.3
Present rotational energy	0.214
Total tidal dissipation in 4.5×10^9 years	~ 2
Total heat loss in 4.5×10^9 years	~ 14.4

Rotational energy and its dissipation by tidal friction, assuming a 6- to 8-h primeval rotation period, are included in Table 4 for comparison, but do not contribute significantly to internal heat. Most of the present tidal dissipation is attributed to the sea and this is presumed to have been the case for much of the life of the Earth.

Ignoring for the moment the final item in Table 4, we can compare the present stored heat (13.3×10^{30} J) with the sum of radiogenic heat and total core formation energy (21.7×10^{30} J) to allow a net heat loss of 8.4×10^{30} J. This is comparable to the heat loss that would have occurred over 4.5×10^9 years if the present rate, 44×10^{12} W (Pollack et al., 1993) had prevailed for the whole of that time. To allow for the early more rapid heat loss, which we estimate to give an average heat flux 2.3 times the present value, yielding the final entry in Table 4, we must appeal to the first entry in Table 4: the Earth started hot because of the accretion energy. With these numbers the total energy budget (stored plus lost heat), 27.7×10^{30} J, must be balanced by 19.9 $\times 10^{30}$ J of gravitational energy and only 7.8×10^{30} J of radiogenic heat. The ratio emphasises that gravitational energy is crucial to the Earth's early thermal history and that, far from being responsible for the Earth's internal heat, radioactivity accounts for less than 60% of the present stored heat and provides only a topping-up.

Our argument assumes that early short-lived radioactivity was not important. The isotopes that appear to be the most serious contenders for early heating are ^{26}Al and ^{60}Fe, but they have half lives of 7.2×10^5 years and 3×10^5 years respectively, less than 1% of the usually estimated interval between nuclear synthesis and earth accretion, 100 to 200 million years. This interval would need to be drastically shortened to bring short-lived radioactivity into serious contention.

5. Gravitational contributions to the energy of an evolving core

Comparison of the results in Table 2 for models B to F allows separate identification of the different contributions to core energy that arise gravitationally from development of the inner core. They are summarised in Table 5 in a form that is readily rescaled to any other assumed density changes. The gravitational energy release and strain energy are obtained directly from the core energies in Table 2, in order down the list (C–B), (D–C), (F–D) and (E–D), except that the (F–D) difference is divided by 10 to correspond to the 66 K core cooling required for inner core formation, instead of 660 K as implied by the 1% density increment in model F. The energy of compression by the mantle is calculated as $\bar{P}\Delta V$, where $\bar{P}$ is the average of the core–mantle boundary pressures in the two models that are compared and ΔV is the volume decrease of the core.

The last line of Table 5 draws attention to the balance of core energies resulting from contraction of the mantle, but it is not strictly correct to infer that the mantle has no effect on core energetics. In these calculations the inner core was constrained to the same mass in models C to F, whereas increasing pressure on the core by mantle contraction raises the melting point faster than the temperature rise by adiabatic compression, causing the inner core to grow. But the consequent net release of core energy is accounted for by inner core development in mod-

Table 5
Core energy changes by four processes. All values in units of 10^{28} J

	Gravitational energy release	Compression by mantle	Strain energy	Net gravitational energy release	Non-gravitational heat release	Adiabatic heating
Compositional separation of inner core[a]	3.979	0.149	0.447	3.681	–	0.294
Freezing of inner core[b]	2.201	1.675	0.267	3.608	6.1	0.230
Core $\Delta\rho/\rho = 0.001$ ($\Delta T \approx -66$ K)	1.902	3.013	0.283	4.632	9.0	0.265
Mantle $\Delta\rho/\rho = 0.01$ ($\Delta T \approx -500$ K)	2.173	3.930	6.103	0	–	1.562
			Total	11.921	15.1	

[a]Assumes a compositional density contrast of 457 kg m^{-3}.
[b]Assumes a solidification density increase of 140 kg m^{-3}.

els B → C → D. It is necessary to remember that in Table 5 we have separated, for the purpose of calculations, processes that occur simultaneously in the Earth. The general rise in pressure accompanying mantle cooling reduces slightly the temperature drop causing inner core development (66 K in these calculations), but by less than the uncertainty in this number.

It is a feature of our results that the mantle makes an important contribution to core energy for those core processes that cause the mantle to contract (the second column of Table 5). Separate identification of the core energy release is necessary because the global energy release includes energy released in the mantle, even for processes that are due to changes in the core.

The third line in Table 5 is adjusted to match the temperature drop in the core that accompanies inner core formation. This is estimated from the difference between the pressure dependence of melting point by Eq. (6) and the adiabatic variation that is assumed to prevail in the liquid with no inner core (model B)

$$(\partial T/\partial P)_S = \gamma T/K_S \tag{20}$$

Using the relationship between isothermal and adiabatic incompressibilities, $K_T = K_S/(1+\gamma\alpha T)$, we have the pressure dependence of the difference between melting point T_M and temperature T_S on an adiabat at the point where T_M and T_S coincide

$$\frac{d}{dP}(T_M - T_S) = \frac{T}{K_S}\left[\gamma - \frac{2}{3} + 2\gamma\alpha T\left(\gamma - \frac{1}{3}\right)\right] \tag{21}$$

With a recent revision of estimated thermodynamic properties of the core (Stacey, 1994), $d(T_M - T_S)/dP = 2.22$ K GPa^{-1} at inner core pressure. With the 30 GPa pressure difference between the centre of the Earth with a fully liquid core (model B) and the inner core boundary in model D, the temperature drop required for the inner core boundary to expand to its present position from the centre of the Earth is 66 K. Then with a volume averaged thermal expansion coefficient of the whole core $\bar{\alpha} = 15.0 \times 10^{-6}$ K^{-1}, the accompanying increase in core density is $\Delta\rho/\rho = 0.0010$, as assumed in Table 5.

6. Discussion

There are two aspects to the conclusions that can be drawn from Table 5: the importance of the sum of the net gravitational energy releases, 11.921×10^{28} J, to the total energy balance of the core and the magnitude of the compositional separation energy, 3.681×10^{28} J, as an effective energy source for the dynamo. Only the compositional separation energy can drive the dynamo with high efficiency. The gravitational and compressional energy released in the process of contraction by freezing and cooling contributes only with the modest thermodynamic efficiency of thermal energy. A simple test to apply in making the distinction is to ask whether a particular process could, in principle, be reversed simply by applying heat to the core. If so then mechanical energy can be derived from it only with the thermodynamic efficiency of a heat engine. This applies not merely to freezing and cooling, but to the gravitational energies in Table 5 that derive from them.

The numbers may be seen in perspective by comparing them with the thermal energies of freezing and cooling. Taking, as an approximation, total core cooling by $\Delta T = 56$ K (since cooling of an adiabat is proportional to temperature, which is lower in the outer part of the core than at the inner core boundary) with specific heat $C_P = 825$ J kg^{-1} K^{-1} and mass $M_C = 1.94 \times 10^{24}$ kg, the heat released by cooling is

$$Q_1 = M_C C_P \Delta T = 9.0 \times 10^{28} \text{ J} \tag{22}$$

The latent heat of freezing of the inner core, Q_2, can be estimated from the entropy of melting by Eq. (8). The total volume change of the inner core resulting from freezing, $\Delta V = 9.97 \times 10^{16}$ m^3, is obtained as the difference between inner core volumes in models C and D and the number of moles in the inner core, n. Assuming a mean atomic weight of 52, intermediate between pure iron and the outer core, $n = 1.9 \times 10^{24}$. Then with $T = 5000$ K

$$Q_2 = T\Delta S = 6.1 \times 10^{28} \text{ J} \tag{23}$$

With these numbers the gravitational energy from Table 5 contributes more than 40% of the total heat lost by the core in the process of inner core formation, 27.0×10^{28} J.

Now we can re-examine the age of the inner core. Rapid stirring of the outer core ensures that it supports an adiabatic temperature gradient and by estimating the thermal conductivity to be $\kappa = 28$ W m^{-1} K^{-1}, Stacey (1992) obtained a conducted heat flux in the outer part of the core of 3.7×10^{12} W. While the value of κ is insecure it is unlikely to be in error by 50%. Also it is not necessary that the core heat loss should equal or exceed the conducted heat because, as Loper (1978b) pointed out, compositional convection may carry heat downwards (refrigerator action). So, taking 3.7×10^{12} W as rough measure of the rate of core heat loss, we find the age of the inner core to be 27.0×10^{28} J$/3.7 \times 10^{12}$ W $= 2.3 \times 10^9$ years. Thus it appears unlikely that the inner core has existed for appreciably more than half of the age of the Earth.

If we take the age of the inner core to be, in fact, 2.3×10^9 years, then the average power of compositionally driven convection over this period is 5×10^{11} W. This appears to be quite adequate for the dynamo and may allow for some refrigerator action, which would increase the estimated inner core age and reduce the need to consider a relatively inefficient thermal convective dynamo in the early earth. However, to take this discussion further we need an analysis of the radial distribution of core energy dissipation.

The energies calculated in Table 5 are proportional to the density differences assumed and can be rescaled with any improved density estimates. The density–pressure relationships used (Table 1) are accurate enough to ensure that no appreciable error results from them. We have made an assumption that needs to be noted. Densities of the liquid inner and outer cores (model C) are related to the density of the homogeneous whole core (model B) by the hypothesis of ideal mixing (Eq. (12)). While this is almost certainly a good approximation for two liquid iron alloys with different solute concentrations, it cannot be exact. We would expect the homogeneous liquid to be slightly more dense than by Eq. (12) and therefore that the gravitational energy of compositional separation is slightly overestimated. But we can see that the effect must be very small by noting that the core volume change accompanying compositional separation is very small and the compressional energy in Table 5 (0.149×10^{28} J) is correspondingly small. A small change to this number would have no influence on our conclusions.

A conclusion that surprised us is that the radius of the homogeneous model (A) is 7 km less than the radius of the Earth with a separated core (model B), although the central pressure is only 60% of the pressure after core separation. We had supposed that the strong increase in central compression by core formation would cause contraction of the whole Earth, but this effect is more than compensated by the steeper increase in pressure in the outer layers. Apart from the generally greater compression of the shallow layers, the depths to silicate phase transitions are less in the homogenised model and the proportion of silicate in high pressure (lower mantle) forms is higher, both for this reason and because the low pressure silicate is diluted by alloy.

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