Pythagoras, the metric tensor and relativity[1]

Pythagoras[2] is regarded to be the first pure mathematician. His famous theorem, known to every student, is the basis for a remarkable thread of geometry that leads directly to Einstein's[3] Theory of Relativity.

1 Pythagoras' Theorem

The sum of the squares of the lengths of the two normal sides of a right triangle equals the square of the length of its hypoteneuse.

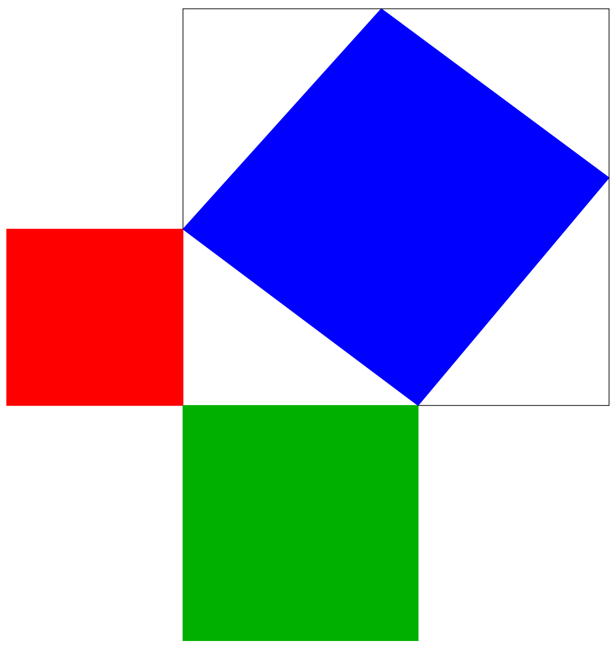

Figure 1: A graphical proof of Pythagoras Theorem

Pythagoras developed the theorem in description of mapping in the *plane geometry* that was extensively explored more than a century later by Euclid of Alexandria[4]. The simple *flat geometry* that so nearly accords with the spatial geometry in which we live is called *Euclidean Geometry* in recognition.

Algebraically, Pythagoras theorem describes the distance between two points in Euclidean 2- or 3-space. Consider 2 vectors $\vec{\boldsymbol{r}}_\mathbf{1}$ and $\vec{\boldsymbol{r}}_\mathbf{2}$, and let $\vec{\boldsymbol{r}}_\mathbf{3} = \vec{\boldsymbol{r}}_\mathbf{2} - \vec{\boldsymbol{r}}_\mathbf{1}$. Pythagoras theorem

[1] ©Olivia Jensen, McGill University

[2] Pythagoras of Samos

[3] Albert Einstein

[4] Euclid of Alexandria

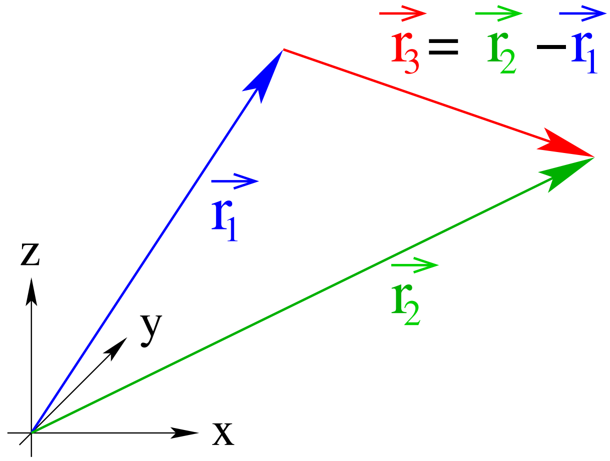

Figure 2: Distance between two vectors

allows us to determine the *length* of the vector $\vec{r}_3$.

$$\ell\{\vec{r}_3\} = |\vec{r}_3| = |\vec{r}_2 - \vec{r}_1|.$$

If the components of the prototypical vector $\vec{r} : [\, r_x, \, r_y, \, r_z \,]$, then Pythagoras theorem tells us that

$$\begin{aligned} \ell^2\{\vec{r}_3\} &= r_{3_x}^2 + r_{3_y}^2 + r_{3_z}^2, \\ &= (r_{2_x} - r_{1_x})^2 + (r_{2_y} - r_{1_y})^2 + (r_{2_y} - r_{1_y})^2. \end{aligned}$$

While it might not look like a simplification of the description, let us rewrite this latter form as

$$\Delta\ell^2 = \Delta x \Delta x + \Delta y \Delta y = \Delta z \Delta z$$

and then in linear algebraic form as

$$\Delta\ell^2 = \left[\begin{array}{ccc} \Delta x & \Delta y & \Delta z \end{array}\right] \bullet \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \bullet \left[\begin{array}{c} \Delta x \\ \Delta y \\ \Delta z \end{array}\right].$$

The $\mathbf{3} \times \mathbf{3}$ *identity matrix* in this form is the prototype of the *metric*, the "measurement rule", for calculating squared lengths via Pythagoras' Theorem in a Euclidean $\mathbf{3}-$space. It practically tells us how the spatial geometry is involved in length measurement. Here the rule is extremely simple because of our choice of coordinates, *Cartesian*, and the geometry of the space, *Euclidean*.

While it might not be clear, this form can be economically written in a tensor form:

$$\Delta\ell^2 = \delta_{ij} \Delta x_i \Delta x_j$$

where, now $\Delta x_i, \Delta x_j$ represent each of $\Delta x, \, \Delta y, \, \Delta z$ as necessary. Note the summation is implied in accord with the Einstein convention. Written fully,

$$\Delta\ell^2 = \sum_{\substack{i=x,y,z \\ j=x,y,z}} \delta_{ij} \Delta x_i \Delta x_j$$

In this description, δ_{ij} is the *metric tensor* of a *Euclidean* $\mathbf{3}$*-space* in *Cartesian coordinates.* Under these special conditions, this rule holds *globally*; that is, we can measure the squared distance, no matter how large, between any two vectors in a *flat space.* While it might not be obvious without reminder, whatever orientation or origin we use for our coordinate system, $\Delta\ell$ remains *invariant.* That is, whatever our perspective, the length of the vector $\vec{r}_3$ is seen to be the same.

We can extend this Pythagorean rule to hold in more general conditions where, for example, space might only be flat locally. We let

$$d\ell^2 = \delta_{ij} dx_i dx_j.$$

If the space is still globally flat, we can integrate $d\ell$ along a straight-line path from $\vec{r}_1$ to $\vec{r}_2$ to obtain $\Delta\ell$ as

$$\Delta\ell = \int_{\vec{r}_1}^{\vec{r}_2} d\ell = \int_{\vec{r}_1}^{\vec{r}_2} \sqrt{\delta_{ij} dx_i dx_j}.$$

If the space is not globally flat while still being locally flat, the shortest path from $\vec{r}_1$ to $\vec{r}_2$ may not be a simple straight line and so we would have to find the shortest path by an extremal principle:

$$\frac{d\Delta\ell}{d\,(\mathbf{path})} = 0.$$

An example of one such extremal path is that taken by a photon as it transits the universe. This path is called the *null geodesic* path. Everywhere. locally, along that path, the photon travels with its speed of light in a straight line. The photon always feels itself to be travelling in a straight line but its straight line might look seriously curved to us from our relatively static perspective as the photon is accelerated towards large masses. Remarkably, a photon may take many paths of exactly equal "length" from a source to a detector – the effect is seen as *gravitational lensing*[5] when light is bent by strong gravitational fields.

1.1 Special relativity

Contributing an elegant geometrical description to Einstein's development of *Special Relativity* theory, Lorentz and Minkowski introduced a non-Euclidean space: *Lorentz-Minkowski space.* Using, still, Cartesian coordinates, the metric for this space is defined and ordered as

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The new *dimension* implicitly introduced relates to the passage of *time*; μ, ν take on the directional components $x_0 = ct$, $x_1 = x\, x_2 = y$ and $x_3 = z$ using the conventional numerical index to represent each coordinate and where, now, c is the constant speed of

[5] Gravitational lensing; note the wispy halos centred on the massive galaxies.

light in free space[6]. The differential squared interval is determined, then, as

$$ds^2 = \eta_{\mu\nu} dx_\mu dx_\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2.$$

Note that eliminating the first row and first column of $\eta_{\mu\nu}$, we are left with the metric tensor of a Euclidean $\mathbf{3}-$space. Lorentz-Minkowski space includes a "distance" scaling for time as well.

Normally, in special relativity, a vector in the Lorentz-Minkowski $\mathbf{4}-$*space-time* is not considered to locate a position; it locates an *event*. An event has a position and a time. The "distance" between events is called the *interval*.

The fundamental statement of Special Relativity from which all details of the theory derive: *The interval between two events in Lorentz-Minkowski space is invariant for all observers.* That is, just as the distance between vectors in Euclidean space is constant for all observers, whatever their origin place or orientation, the *interval* is seen to be the same by all observers in Lorentz-Minkowski space, whatever their orientation, place or *velocity*. Einstein's grand theory of *Gravitation*, colloquially called *General Relativity* extended this fundamental statement's condition to also include ... *whatever their acceleration whether inertial or gravitational*.

1.2 Einstein's Gravitation – General Relativity

In General Relativity, the metric tensor can become almost arbitrarily complex. It is generally described as

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix}.$$

Each of its elements depend upon place and the coordinate system employed in addressing the place. The elements of the metric tensor include all information about the *geometry* at place and time.

If one is not under relativistic accelerations, either inertial or gravitational, the metric tensor of gravitation theory is typically not substantially different from that of empty Lorentz-Minkowski space. One can often apply a *perturbation theory* in its description as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

Where we stand on the surface of a non-rotating spherical Earth (which has far too little mass to offer us a relativistic acceleration), using a coordinate system with the $x_3 = z-$coordinate oriented radially,

[6] Now, in rationalizing the SI system of units, the speed of light in free space is defined to be *exactly* $\mathbf{299792458 m\,s^{-1}}$; the *metre* now is defined to be $\mathbf{1/299792458}$ the distance travelled by light in exactly one second; the *second* is defined to be the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom. The third fundamental unit of the SI system of measures is the *kilogram* and it is still defined by the "international prototype" held at the Bureau International des Poids et Measures in Paris. The kilogram.

$$h_{\mu\nu} = \begin{bmatrix} -\frac{2\Phi}{c^2} & \frac{\sqrt{2\Phi^3}}{c^3} & \frac{\sqrt{2\Phi^3}}{c^3} & \frac{\sqrt{2\Phi^3}}{c^3} \\ \frac{\sqrt{2\Phi^3}}{c^3} & -\frac{2\Phi}{c^2} & \frac{2\Phi^2}{c^4} & \frac{2\Phi^2}{c^4} \\ \frac{\sqrt{2\Phi^3}}{c^3} & \frac{2\Phi^2}{c^4} & -\frac{2\Phi}{c^2} & \frac{2\Phi^2}{c^4} \\ \frac{\sqrt{2\Phi^3}}{c^3} & \frac{2\Phi^2}{c^4} & \frac{2\Phi^2}{c^4} & -\frac{2\Phi}{c^2} \end{bmatrix}.$$

Here, Φ is the *gravitational potential* at the surface of the Earth

$$\Phi(r_s) = \frac{-GM_\oplus}{r_s}.$$

The g_{00} element describes the slight, $\sim 2.6 \times 10^{-5}$, temporal *gravitational redshift* relative to $r \longrightarrow \infty$, due to the attraction of the Earth's mass on photons and the g_{03} and g_{30} elements describe a slight spatial dilation of wavelengths in the x_3, radial direction.

General relativistic gravitational waves as might be produced by supernoval explosion and collapse or by massive stars or black holes in very close orbits produce temporal variations in the purely spatial and off-diagonal elements of the metric tensor. These extremely small temporal variations can be described as

$$h_{ij}(\vec{r}, t) \sim h_{ij} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

where $\vec{k}$ is the propagation vector direction radially from the source. These waves travel with the velocity of light. The perturbation metric of the geometric distortion of space as the wave travels by has form

$$h_{\mu\nu}(\vec{r}, t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_{12} & h_{13} \\ 0 & h_{12} & 0 & h_{23} \\ 0 & h_{13} & h_{23} & 0 \end{bmatrix} e^{-i(\vec{k}\cdot\vec{r} - \omega t)}.$$

The LIGO[7] (Laser Interferometry Gravitational-Wave Observatory) has been constructed at a cost of US$394 million to search for gravitational waves that might be caused by violent astrophysical events. The experiment was initially designed for an eventual sensitivity of $|h_{ij}| \sim 10^{-23}$ for frequencies of oscillations between about $1\mathrm{kHz}$ and $10\mathrm{kHz}$. The first-stage interferometers were built and calibration test runs completed during 6 years. Nothing was detected! The "Advanced LIGO" upgrade was completed in September 2015 at a further cost of US$620 million. It now approaches the design sensitivity ($|h_{ij}| \sim 10^{-23}$) for frequencies between $300\mathrm{Hz}$ and $3\mathrm{kHz}$.[8] The LIGO proponents expect to see between 1 and 3 events per year caused by neutron star and black hole binary systems and spin of asymmetric neutron stars. LIGO announced detection of gravitational waves on February 11, 2016.

[7] LIGO Scientific Collaboration.

[8] Gravitational wave spectrum